

GENERIC EXPONENTIAL SUMS ASSOCIATED TO LAURENT POLYNOMIALS IN ONE VARIABLE

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Abstract. The generic Newton polygons for L -functions of exponential sums associated to Laurent polynomials in one variable are determined when p is large. The corresponding Hasse polynomials are also determined.

Key words: exponential sum, L -function, generic Newton polygon

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1. INTRODUCTION

We shall determine the generic Newton polygon of L -functions of exponential sums associated to Laurent polynomials in one variable.

Throughout this paper, p denotes a prime number, and q denotes a power of p . Write $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$. Let $\bar{\mathbb{F}}_p$ be a fixed algebraic closure of the finite field \mathbb{F}_p , and \mathbb{F}_q the finite field with q elements in $\bar{\mathbb{F}}_p$.

Let f be a Laurent polynomial over \mathbb{F}_q . We assume that the leading exponents of f are prime to p . One associates to f the Artin-Schreier curve

$$C_f : y^q - y = f(x).$$

Let N_k be the number of \mathbb{F}_{q^k} -rational points including the infinities on C_f . The zeta function of C_f is defined by

$$Z(t, C_f, \mathbb{F}_q) = \exp\left(\sum_{k=1}^{+\infty} N_k \frac{t^k}{k}\right).$$

Let $\bar{\mathbb{Q}}$ be a fixed algebraic closure of \mathbb{Q} . Let ψ denote any nontrivial character of \mathbb{F}_p into $\bar{\mathbb{Q}}^\times$. Let V_f be the affine line \mathbb{A} over \mathbb{F}_q if f is a polynomial, and let V_f be the one-dimensional torus \mathbb{T} over \mathbb{F}_q if f is not a polynomial. We have

$$N_k = q^k + 1 + \sum_{\alpha \in \mathbb{F}_q^\times} S(k, \alpha f, \mathbb{F}_q),$$

where the exponential sum $S(k, f, \mathbb{F}_q)$ is defined by

$$S(k, f, \mathbb{F}_q) = \sum_{x \in V_f(\mathbb{F}_{q^k})} \psi(\text{Tr}_{\mathbb{F}_{q^k}/\mathbb{F}_p}(f(x))).$$

So we have

$$(1-t)(1-qt)Z(t, C_f, \mathbb{F}_q) = \prod_{\alpha \in \mathbb{F}_q^\times} L(t, \alpha f, \mathbb{F}_q),$$

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where the L -function $L(t, f, \mathbb{F}_q)$ is defined by

$$L(t, f, \mathbb{F}_q) = \exp\left(\sum_{k=1}^{+\infty} S(k, f, \mathbb{F}_q) \frac{t^k}{k}\right).$$

It is well-known that the function $L(t, f, \mathbb{F}_q)$ is a polynomial in t with coefficients in $\overline{\mathbb{Q}}$.

Let \mathbb{Z}_p be the ring of p -adic integers, and \mathbb{Q}_p the field of p -adic numbers. Fix an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}_p}$. Let $\text{ord}_p(\cdot)$ be the p -adic order function of $\overline{\mathbb{Q}_p}$, and define the q -adic order function as $\text{ord}_q(\cdot) = \frac{1}{\text{ord}_p(q)} \text{ord}_p(\cdot)$. As $L(t, f, \mathbb{F}_q)$ has coefficients in $\overline{\mathbb{Q}}$, one can talk about the p -adic absolute values of its reciprocal roots. These p -adic absolute values are completely determined by the Newton polygon of $L(t, f, \mathbb{F}_q)$ defined as follows.

Definition 1.1. Let $g(t) = 1 + \sum_{i=1}^u c_i t^i$ be a polynomial in t with coefficients $c_i \in \overline{\mathbb{Q}_p}$. The q -adic Newton polygon of g is the lower convex closure of the points

$$(0, 0), (n, \text{ord}_q(c_n)), \quad n = 1, \dots, u.$$

It is very hard to determine the Newton polygon of $L(t, f, \mathbb{F}_q)$ in general. However, it is easier to give a good lower bound. The simplest one is the Hodge polygon defined as follows.

Definition 1.2. Let d be a positive integer. The Hodge polygon of the interval $[0, d]$ is the polygon whose vertices are

$$\left(n, \frac{n(n+1)}{2d}\right), \quad n = 0, 1, 2, \dots, d-1.$$

Definition 1.3. Let d and e be positive integers. The Hodge polygon of $[-e, d]$ is the polygon with initial point $(0, 0)$, end point $(d+e, \frac{d+e}{2})$, and the vertices $(m+n+1, \frac{m(m+1)}{2e} + \frac{n(n+1)}{2d})$ with (m, n) running over pairs satisfying

$$-\frac{1}{e} < \frac{m}{e} - \frac{n}{d} < \frac{1}{d}, \quad 0 \leq m < e, 0 \leq n < d.$$

Let $\Delta(f)$ be the smallest closed interval of the real line containing 0 and the exponents of the monomials of f . So $\Delta(f) = [0, d]$ if f is a polynomial of degree d , and $\Delta(f) = [-e, d]$ if f is a Laurent polynomial with leading term $a_{-e}x^{-e} + a_d x^d$. The well-known Hodge bound is stated as the following theorem.

Theorem 1.4 (Hodge bound). *The q -adic Newton polygon of $L(t, f, \mathbb{F}_q)$ lies above the Hodge polygon of $\Delta(f)$. Moreover, both polygons have the same initial point and the same end point.*

By Grothendieck's specialization lemma (Confer [K76] and [W04]), the q -adic Newton polygon of $L(t, f, \mathbb{F}_q)$ is constant for a generic f with fixed $\Delta(f) = \Delta$. That constant polygon is called the generic Newton polygon of Δ . Let $d > 0$ and $e \geq 0$ be integers. Let $D = d$ if $e = 0$, and let D be the least common multiple of d and e if $e > 0$. We assume that D is prime to p . The following theorem says that in nice situations the generic polygon coincides with the Hodge polygon.

Theorem 1.5 (Stickelberger's theorem [W93]). *The generic Newton polygon of $[-e, d]$ coincides with its Hodge polygon if and only if $p \equiv 1 \pmod{D}$.*

As a special case of a conjecture of Wan [W04], the following theorem says that the Hodge bound is approximately the best.

Theorem 1.6 (Zhu [Zh03, Zh04, Zhu04]). *The generic Newton polygon of $[-e, d]$ goes to its Hodge polygon as p goes to infinity.*

In proving the above theorem in the case $e = 0$, Zhu used Dwork's p -adic theory, a kind of Diédonne-Manin diagonalization, and some force computations to produce a list of polynomials she denoted as f_t 's. She then used a kind of maximal-monomial-locating technique to prove that one of these f_t 's does not vanish. Blache-Férard [BF] discovered that Zhu's maximal-monomial-locating technique can prove the nonvanishing of f_0 . This enabled them to get the following theorem.

Theorem 1.7 (Blache-Férard). *If $p \geq 3D$, the generic Newton polygon of $[0, d]$ is the polygon with vertices*

$$(n, \frac{1}{p-1} \sum_{i=1}^n \lceil \frac{pi-n}{d} \rceil), \quad n = 0, 1, \dots, d-1.$$

The condition $p \geq 3D$ in the theorem is very clean. To achieve that clean condition Blache-Férard abolished Zhu's Diédonne-Manin diagonalization technique, and made recourse to Dwork's original method.

It should be mentioned that Yang [Ya] computed the Newton polygons for L -function of exponential sums associated to polynomials of the form $x^d + \lambda x$, and Hong [H01, H02] computed the Newton polygons for L -function of exponential sums associated to polynomials of degree 4 and 6.

From now on we assume that $e > 0$. We shall determine the generic Newton polygon of $[-e, d]$.

Write

$$p_{[0,d]}(n) = \frac{1}{p-1} \sum_{i=1}^n \lceil \frac{pi-n}{d} \rceil, \quad n = 0, 1, \dots, d.$$

And write

$$p_{[-e,d]}(0) = 0, p_{[-e,d]}(d+e) = \frac{d+e}{2},$$

$$p_{[-e,d]}(k) = \min_{(m,n) \in I_k} \{p_{[0,e]}(m) + p_{[0,d]}(n)\}, \quad k = 1, \dots, d+e-1,$$

where

$$I_k = \{(m, n) \mid m + n + 1 = k, -\frac{1}{e} \leq \frac{m}{e} - \frac{n}{d} \leq \frac{1}{d}, 0 \leq m < e, 0 \leq n < d\}.$$

Definition 1.8. *The arithmetic polygon of $[-e, d]$ is defined to be the graph of the function on $[0, d+e]$ which is linear between consecutive integers and takes on the value $p_{[-e,d]}(k)$ at integers $k = 0, 1, \dots, d+e$.*

We shall prove the following theorem.

Theorem 1.9. *The generic Newton polygon of $[-e, d]$ coincides with its arithmetic polygon if $p \geq 3D$.*

It would be interesting if one can extend the result to twisted exponential sums and to exponential sums associated to functions studied in [Zhu04] and [BFZ].

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2. THE ARITHMETIC POLYGON

Recall that, for $k = 1, \dots, d + e - 1$,

$$I_k = \{(m, n) \mid m + n + 1 = k, -\frac{1}{e} \leq \frac{m}{e} - \frac{n}{d} \leq \frac{1}{d}, 0 \leq m < e, 0 \leq n < d\}.$$

Let V_k be the subset of I_k consisting pairs at which the function

$$(m, n) \mapsto p_{[0,e]}(m) + p_{[0,d]}(n)$$

takes on the minimal value. In this section we shall prove the following theorem.

Theorem 2.1. *Let $p > 3D$. Then the arithmetic polygon of $[-e, d]$ is convex. Moreover, $(k, p_{[-e,d]}(k))$ ($0 < k < d + e$) is a vertex if and only if V_k contains only one pair.*

We begin with the following lemma.

Lemma 2.2. *The set I_k contains one or two pairs. If $I_k = \{(m, n)\}$, then*

$$-\frac{1}{e} < \frac{m}{e} - \frac{n}{d} < \frac{1}{d}.$$

If I_k contains exactly two pairs, then it is of form $\{(m, n), (m + 1, n - 1)\}$ with

$$\frac{m + 1}{e} = \frac{n}{d}.$$

Proof. Define a degree function on \mathbb{Z} by

$$\deg(i) = \begin{cases} i/d, & i \geq 0, \\ -i/e, & i \leq 0. \end{cases}$$

There is a positive integer u such that

$$k = \#\{i \in \mathbb{Z} \mid \deg(i) \leq u/D\},$$

or

$$\#\{i \in \mathbb{Z} \mid \deg(i) \leq u/D\} < k < \#\{i \in \mathbb{Z} \mid \deg(i) \leq (u + 1)/D\}.$$

If $k = \#\{i \in \mathbb{Z} \mid \deg(i) \leq u/D\}$, then I_k is of form $\{(m, n)\}$ with

$$-\frac{1}{e} < \frac{m}{e} - \frac{n}{d} < \frac{1}{d}.$$

If $\#\{i \in \mathbb{Z} \mid \deg(i) \leq u/D\} < k < \#\{i \in \mathbb{Z} \mid \deg(i) \leq (u + 1)/D\}$, then I_k is of form $\{(m, n), (m + 1, n - 1)\}$ with

$$\frac{m + 1}{e} = \frac{n}{d}.$$

The lemma is proved.

It is easy to see that Theorem 2.1 follows from the following three theorems.

Theorem 2.3. *Let $k = 1, 2, \dots, d + e - 1$. If V_k contains two pairs, then*

$$2p_{[-e,d]}(k) = p_{[-e,d]}(k-1) + p_{[-e,d]}(k+1).$$

Theorem 2.4. *Let $k = 1, 2, \dots, d + e - 1$. If I_k contains two pairs but V_k contains only one pair, then*

$$2p_{[-e,d]}(k) < p_{[-e,d]}(k-1) + p_{[-e,d]}(k+1).$$

Theorem 2.5. *Let $p > 3D$. Let $k = 1, 2, \dots, d + e - 1$. If I_k contains only one pair, then*

$$2p_{[-e,d]}(k) < p_{[-e,d]}(k-1) + p_{[-e,d]}(k+1).$$

Proof of Theorem 2.3. Suppose that V_k contains two pairs. Then so does I_k . Assume that $I_k = \{(m, n), (m+1, n-1)\}$. Then $\frac{m+1}{e} = \frac{n}{d}$. It follows that $I_{k-1} = \{(m, n-1)\}$ and $I_{k+1} = \{(m+1, n)\}$. Note that

$$p_{[-e,d]}(k) = p_{[0,e]}(m) + p_{[0,d]}(n) = p_{[0,e]}(m+1) + p_{[0,d]}(n-1).$$

It follows that

$$2p_{[-e,d]}(k) = p_{[0,e]}(m) + p_{[0,d]}(n) + p_{[0,e]}(m+1) + p_{[0,d]}(n-1) = p_{[-e,d]}(k-1) + p_{[-e,d]}(k+1).$$

Theorem 2.3 is proved.

Proof of Theorem 2.4. Assume that $I_k = \{(m, n), (m+1, n-1)\}$. Then $\frac{m+1}{e} = \frac{n}{d}$. It follows that $I_{k-1} = \{(m, n-1)\}$ and $I_{k+1} = \{(m+1, n)\}$. Without loss of generality, we assume that $V_k = \{(m, n)\}$. Then

$$p_{[0,e]}(m) + p_{[0,d]}(n) < p_{[0,e]}(m+1) + p_{[0,d]}(n-1).$$

It follows that

$$2p_{[0,e]}(m) + 2p_{[0,d]}(n) < p_{[0,e]}(m) + p_{[0,d]}(n) + p_{[0,e]}(m+1) + p_{[0,d]}(n-1).$$

That is,

$$2p_{[-e,d]}(k) < p_{[-e,d]}(k-1) + p_{[-e,d]}(k+1).$$

Theorem 2.4 is proved.

Proof of Theorem 2.5. Assume that $I_k = \{(m, n)\}$. Then

$$-\frac{1}{e} < \frac{m}{e} - \frac{n}{d} < \frac{1}{d}.$$

Let $(m_1, n_1) \in V_{k-1}$. Then $m_1 = m$ or $n_1 = n$. Without loss of generality, we assume that $m_1 = m$. Then $n_1 = n-1$. Let $(m_2, n_2) \in V_{k+1}$. Then $m_2 = m$ or $n_2 = n$.

First, we assume that $m_2 = m$. Then $n_2 = n+1$. Note that

$$p_{[0,d]}(n+1) - p_{[0,d]}(n) \geq \frac{1}{p-1}(\lceil (p-1)\frac{n+1}{d} \rceil - 1),$$

$$p_{[0,d]}(n) - p_{[0,d]}(n-1) \leq \frac{1}{p-1}\lceil (p-1)\frac{n}{d} \rceil,$$

and

$$\lceil (p-1)\frac{n}{d} \rceil < \lceil (p-1)\frac{n+1}{d} \rceil - 1.$$

It follows that

$$2p_{[0,d]}(n) < p_{[0,d]}(n+1) + p_{[0,d]}(n-1).$$

Therefore

$$2p_{[0,e]}(m) + 2p_{[0,d]}(n) = p_{[0,e]}(m) + p_{[0,d]}(n+1) + p_{[0,e]}(m) + p_{[0,d]}(n-1).$$

That is,

$$2p_{[-e,d]}(k) < p_{[-e,d]}(k-1) + p_{[-e,d]}(k+1).$$

Secondly, we assume that $n_2 = n$. Then $m_2 = m+1$. Note that

$$p_{[0,e]}(m+1) - p_{[0,e]}(m) \geq \frac{1}{p-1}(\lceil (p-1)\frac{m+1}{e} \rceil - 1),$$

$$p_{[0,d]}(n) - p_{[0,d]}(n-1) \leq \frac{1}{p-1}\lceil (p-1)\frac{n}{d} \rceil,$$

and

$$\lceil (p-1)\frac{n}{d} \rceil < \lceil (p-1)\frac{m+1}{e} \rceil - 1.$$

It follows that

$$p_{[0,e]}(m) + p_{[0,d]}(n) < p_{[0,e]}(m+1) + p_{[0,d]}(n-1).$$

Therefore

$$2p_{[0,e]}(m) + 2p_{[0,d]}(n) < p_{[0,e]}(m+1) + p_{[0,d]}(n) + p_{[0,e]}(m) + p_{[0,d]}(n-1).$$

That is,

$$2p_{[-e,d]}(k) < p_{[-e,d]}(k-1) + p_{[-e,d]}(k+1).$$

Theorem 2.5 is proved.

3. HASSE POLYNOMIAL

For $\vec{a} = (a_{-e}, \dots, a_d) \in \mathbb{F}_q^{d+e+1}$, we write

$$f_{\vec{a}}(x) = \sum_{i=-e}^d a_i x^i.$$

It is easy to see that the Newton polygon of $L(t, f_{\vec{a}}, \mathbb{F}_q)$ is independent of a_0 . So one can take a_0 to be any preferred number. We take $a_0 = 1$ so that Lemmas 5.2 and 5.3 are expressed in a simpler form.

In this section we define a polynomial H such that the Newton polygon of $L(t, f_{\vec{a}}, \mathbb{F}_q)$ coincides with the generic Newton polygon of $[-e, d]$ if and only if $H(\vec{a}) \neq 0$.

Definition 3.1. Let $k = 1, 2, \dots, d+e-1$ be such that $V_k = \{(m, n)\}$. We define S_k to be the set of permutations τ of $\{-m, -m+1, \dots, n\}$ such that

$$\tau(i) \begin{cases} \geq n - d\lceil -\frac{pi-n}{d} \rceil, & \text{if } i > 0, \\ = 0, & \text{if } i = 0, \\ \leq -m + e\lceil \frac{pi+m}{e} \rceil, & \text{if } i < 0. \end{cases}$$

Let

$$E(t) = \exp\left(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}\right).$$

It is a power series in $\mathbb{Z}_p[[t]]$, and we call it the Artin-Hasse exponential series. We write

$$E(t) = \sum_{n=0}^{+\infty} \lambda_n t^n.$$

Definition 3.2. Let $k = 1, 2, \dots, d + e - 1$ be such that $V_k = \{(m, n)\}$. We write

$$r_i = \begin{cases} n - d\left\{\frac{pi-n}{d}\right\} + d, & 1 \leq i \leq n, \\ m - e\left\{\frac{pi+m}{e}\right\} + e, & -m \leq i \leq -1. \end{cases}$$

We define a polynomial H_k in the variables x_{-e}, \dots, x_d , by

$$H_k(\vec{x}) = \sum_{\tau \in S_k} u_{\tau} \prod_{i=-m}^{-1} x_{-r_i-\tau(i)} \prod_{i=1}^n x_{r_i-\tau(i)},$$

where

$$u_{\tau} = \text{sgn}(\tau) \left(\prod_{i=1}^n \lambda_{\lfloor \frac{pi-\tau(i)}{d} \rfloor} \lambda_{\lceil \frac{pi-\tau(i)}{d} \rceil} \right) \prod_{i=-m}^{-1} \lambda_{\lfloor \frac{-pi+\tau(i)}{e} \rfloor} \lambda_{\lceil \frac{-pi+\tau(i)}{e} \rceil} \in \mathbb{Z}_p^{\times}.$$

Definition 3.3. The Hasse polynomial H of $[-e, d]$ is defined by

$$H = x_d x_{-e} \prod_{\#V_k=1} \bar{H}_k,$$

where \bar{H}_k is the reduction of H_k modulo p .

We shall prove the following theorem.

Theorem 3.4. The Hasse polynomial H of $[-e, d]$ is non-zero.

Let $k = 1, 2, \dots, d + e - 1$ be such that $V_k = \{(m, n)\}$. It is easy to see that Theorem 3.4 follows from the following one.

Theorem 3.5. Among the monomials

$$\prod_{i=-m}^{-1} x_{-r_i-\tau(i)} \prod_{i=1}^n x_{r_i-\tau(i)}, \quad \tau \in S_k,$$

there is a monomial which appears exactly once.

That theorem plays a crucial role in the determination of the generic Newton polygon of $[-e, d]$. In the case $e = 0$, Blache-Férard [BF] used Zhu's maximal-monomial-locating technique to prove the theorem. In the case $e > 0$, the maximal-monomial-locating technique no longer works. Fortunately, a minimal-monomial-locating technique will play the role.

Set $x_1 < x_2 < \dots < x_d$ and $x_{-1} < x_{-2} < \dots < x_{-e}$. Define $\prod_{i \in I} x_i > \prod_{j \in J} x_j$ and $\prod_{i \in I} x_{-i} > \prod_{j \in J} x_{-j}$ if I and J are finite subsets of positive integers and there is an $i \in I$ which is greater than all $j \in J$. Define $g_1 g_3 \geq g_2 g_4$ if g_1, g_2, g_3, g_4 are monomials such that $g_1 \geq g_2$ and $g_3 \geq g_4$.

It is easy to see that Theorem 3.5 follows from the following one.

Theorem 3.6. *Among the monomials*

$$\prod_{i=-m}^{-1} x_{-r_i-\tau(i)} \prod_{i=1}^n x_{r_i-\tau(i)}, \quad \tau \in S_k,$$

the minimal monomial appears exactly once.

Proof. Note that $r_i \neq r_j$ and $r_{-i} \neq r_{-j}$ if i and j are distinct positive integers. So we can order them such that

$$r_{i_1} > r_{i_2} > \cdots > r_{i_n}, \quad i_j > 0,$$

and

$$r_{t_1} > r_{t_2} > \cdots > r_{t_m}, \quad t_j < 0.$$

Note that $r_{i_1} \leq n + d$ and $r_{t_1} \leq m + e$. So we have

$$r_{i_j} \leq n + d + 1 - j, \text{ and } r_{t_j} \leq m + e + 1 - j.$$

Recall that $\tau \in S_k$ if and only if $\tau(i) \geq r_i - d$ if $i > 0$, and $\tau(i) \leq -r_i + e$ if $i < 0$. Hence, if we define τ_0 by

$$\tau_0(i_j) = n + 1 - j, \text{ and } \tau_0(t_j) = -(m + 1 - j),$$

then $\tau_0 \in S_k$.

We claim that, for any $\tau \in S_k$,

$$\prod_{j=1}^n x_{r_{i_j}-\tau(i_j)} \geq \prod_{j=1}^n x_{r_{i_j}-(n+1-j)}$$

with equality holding if and only if $\tau(i_j) = n + 1 - j$ for all $1 \leq j \leq n$. Suppose that $\tau(i_j) \neq n + 1 - j$ for some $1 \leq j \leq n$. Let j_0 be the least one with this property. Then $\tau(i_{j_0}) < n + 1 - j_0$. Hence

$$r_{i_{j_0}} - \tau(i_{j_0}) > r_{i_{j_0}} - (n + 1 - j_0) \geq r_{i_j} - (n + 1 - j), \text{ for all } j \geq j_0.$$

Therefore

$$\prod_{j=1}^n x_{r_{i_j}-\tau(i_j)} > \prod_{j=1}^n x_{r_{i_j}-(n+1-j)}$$

as claimed.

Similarly, we can prove that, for any $\tau \in S_k$,

$$\prod_{j=1}^m x_{-r_{t_j}-\tau(t_j)} \geq \prod_{j=1}^m x_{-r_{t_j}+(m+1-j)}$$

with equality holding if and only if $\tau(t_j) = -(m + 1 - j)$ for all $1 \leq j \leq m$. It follows that the monomial

$$\prod_{j=1}^n x_{r_{i_j}-(n+1-j)} \prod_{j=1}^m x_{-r_{t_j}+(m+1-j)}$$

is minimal and occurs in the monomials

$$\prod_{i=-m}^{-1} x_{-r_i-\tau(i)} \prod_{i=1}^n x_{r_i-\tau(i)}, \quad \tau \in S_k.$$

The theorem is proved.

4. DWORK'S p -ADIC ANALYTIC METHOD

In this section we give a brief survey on Dwork's p -adic analytic method. Proofs of theorems in this section may be omitted. Interested readers may consult [Dw62, Dw64] and [AS87, AS89] for detailed proofs.

Write $\mathbb{Z}_q := \mathbb{Z}_p[\mu_{q-1}]$ and $\mathbb{Q}_q := \mathbb{Q}_p(\mu_{q-1})$, where μ_n is the group of n -th roots of unity.

Recall that

$$E(t) = \exp\left(\sum_{i=0}^{\infty} \frac{t^{p^i}}{p^i}\right) = \sum_{n=0}^{+\infty} \lambda_n t^n \in \mathbb{Z}_p[[t]]$$

is the Artin-Hasse exponential series. Choose $\pi \in \mathbb{Q}_p(\mu_p)$ such that $E(\pi) = \psi(1)$. We have $\text{ord}_p(\pi) = \frac{1}{p-1}$ and $\sum_{i=0}^{\infty} \frac{\pi^{p^i}}{p^i} = 0$.

Let L be the Banach space over $\mathbb{Q}_q[\pi^{1/D}]$ with formal basis $\pi^{\deg(i)} x^i$, $i \in \mathbb{Z}$. That is, $L = L_0 \otimes_{\mathbb{Z}_q} \mathbb{Q}_q$ with

$$L_0 = \left\{ \sum_{i \in \mathbb{Z}} c_i \pi^{\deg(i)} x^i : c_i \in \mathbb{Z}_q[\pi^{1/D}] \right\}.$$

The space is closed under multiplication. So it is an algebra.

For $\vec{a} = (a_{-e}, \dots, a_d) \in \mathbb{F}_q^{d+e+1}$, we write

$$E_{\vec{a}}(x) := \prod_{i=-e}^d E(\pi \hat{a}_i x^i),$$

where \hat{a}_i is the Teichmüller lifting of a_i . As each $E(\pi \hat{a}_i x^i)$ lies in L , so does $E_{\vec{a}}$.

The Galois group $\text{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$ acts on L but keeps $\pi^{1/D}$ and x fixed. Let σ be the Frobenius element of that Galois group. Write

$$\hat{E}_{\vec{a}}(x) = \prod_{j=0}^{+\infty} E_{\vec{a}}^{\sigma^j}(x^{p^j}).$$

Define an operator $\partial : L \rightarrow L$ by

$$\partial(g) = xg'(x) + g(x)x \frac{d \log \hat{E}_{\vec{a}}(x)}{dx}.$$

It is easy to see that L_0 is stable under ∂ .

Define an operator $\psi_p : L \rightarrow L$ by

$$\psi_p\left(\sum_{i \in \mathbb{Z}} c_i x^i\right) = \sum_{i \in \mathbb{Z}} c_{pi} x^i.$$

And write

$$\Psi_p := \sigma^{-1} \circ \psi_p \circ E_{\vec{a}}.$$

That is,

$$\Psi_p(g) = \sigma^{-1}(\psi_p(gE_{\vec{a}})).$$

Note that Ψ_p is $\mathbb{Q}_p[\pi^{\frac{1}{D}}]$ -linear, but $\mathbb{Q}_q[\pi^{\frac{1}{D}}]$ -semi-linear.

Define $\Psi_{p^n} = \Psi_p^n$. So $\Psi_{q^n} = \Psi_q^n$. It is easy to check that Ψ_q is $\mathbb{Z}_q[\pi^{1/D}]$ -linear. Moreover, we have

$$q\partial\Psi_q = \Psi_q\partial.$$

Let $\bar{\Psi}_p$ be the induced operator of Ψ_p on $L/(\partial L)$. We have the following three theorems.

Theorem 4.1. *We have*

$$L(s, f_{\bar{a}}, \mathbb{F}_q) = \det(1 - s\bar{\Psi}_q \mid L/(\partial L) \text{ over } \mathbb{Q}_q(\pi^{1/D})).$$

Theorem 4.2. *The q -adic Newton polygons of $\det(1 - s^b\bar{\Psi}_q \mid L/(\partial L) \text{ over } \mathbb{Q}_q(\pi^{1/D}))$ and $\det(1 - s\bar{\Psi}_p \mid L/(\partial L) \text{ over } \mathbb{Q}_p(\pi^{1/D}))$ coincide.*

Theorem 4.3. *Over $\mathbb{Z}_q[\pi^{1/D}]$, the lattice $L_0/(\partial L_0)$ has a basis represented by*

$$\pi^{\deg(i)} x^i, \quad -e \leq i \leq d-1.$$

5. ELEMENTARY ESTIMATES

In this section we give some elementary estimates on the matrix coefficients of the operator $\bar{\Psi}_p$ on $L/(\partial L)$.

Write

$$E_{\bar{a}}(x) = \sum_{i \in \mathbb{Z}} \gamma_i x^i.$$

We have

$$\gamma_i = \sum_{\sum_{j=-e}^d j n_j = i} \pi^{j=-e} n_j \prod_{j=-e}^d \lambda_{n_j} \hat{a}_j^{n_j}.$$

Definition 5.1. *We write $\alpha = O(\pi^t)$ to mean that $\text{ord}_\pi(\alpha) \geq t$, where $\text{ord}_\pi(\cdot) = \frac{1}{\text{ord}_p(\pi)} \text{ord}_p(\cdot)$.*

Lemma 5.2. *If $i \geq 0$,*

$$\gamma_i = \pi^{\lceil \frac{i}{d} \rceil} \lambda_{\lfloor \frac{i}{d} \rfloor} \lambda_{\lceil \frac{i}{d} \rceil} \hat{a}_d^{\lfloor \frac{i}{d} \rfloor} \hat{a}_{d\{\frac{i}{d}\}} + O(\pi^{\lceil \frac{i}{d} \rceil + 1}).$$

Proof. If $\sum_{j=-e}^d j n_j = i$ ($n_j \geq 0$), then $\sum_{j=-e}^d n_j \geq \lceil \frac{i}{d} \rceil$ with equality holding if and only if

$$n_j = \begin{cases} \lfloor \frac{i}{d} \rfloor, & j = d \\ \lceil \frac{i}{d} \rceil, & j = d\{\frac{i}{d}\} \\ 0, & \text{otherwise.} \end{cases}$$

The lemma now follows.

Similarly, we have the following lemma.

Lemma 5.3. *If $i < 0$,*

$$\gamma_i = \pi^{\lceil \frac{-i}{e} \rceil} \lambda_{\lfloor \frac{-i}{e} \rfloor} \lambda_{\lceil \frac{-i}{e} \rceil} \hat{a}_{-e}^{\lfloor \frac{-i}{e} \rfloor} \hat{a}_{-e\{\frac{-i}{e}\}} + O(\pi^{\lceil \frac{-i}{e} \rceil + 1}).$$

From the last two lemmas we infer the following corollary.

Corollary 5.4. *We have*

$$\gamma_i = O(\pi^{\lceil \deg(i) \rceil}).$$

Let $F = (F_{ij})_{-e \leq i, j \leq d-1}$ be the matrix defined by

$$\psi_p \circ E_{\bar{a}}(x^j) \equiv \sum_{i=-e}^{d-1} F_{ij} x^i \pmod{\partial L}.$$

Lemma 5.5. *Let $p \geq 3D$, and $-e \leq i, j \leq d-1$. We have*

$$F_{ij} = \gamma_{pi-j} + \begin{cases} O(\pi^{\lceil \deg(pi) \rceil + 2}), & i \neq -e \\ O(\pi^p), & i = -e. \end{cases}$$

Proof. We have

$$\psi_p \circ E_{\bar{a}}(x^j) = \sum_{i_0 \in \mathbb{Z}} \gamma_{pi_0-j} x^{i_0} = \sum_{i_0=-e}^{d-1} \gamma_{pi_0-j} x^{i_0} + \sum_{i_0 \notin \{-e, \dots, d-1\}} \gamma_{pi_0-j} x^{i_0}.$$

For $i_0 \notin \{-e, \dots, d-1\}$, we write

$$\pi^{\deg(i_0)} x^{i_0} = \sum_{i=-e}^{d-1} c_{ii_0} \pi^{\deg(i)} x^i \pmod{\partial L}, \quad c_{ii_0} \in \mathbb{Z}_q[\pi^{1/(de)}].$$

Then

$$\psi_p \circ E_{\bar{a}}(x^j) = \sum_{i=-e}^{d-1} x^i (\gamma_{pi-j} + \sum_{i_0 \notin \{-e, \dots, d-1\}} c_{ii_0} \pi^{\deg(i) - \deg(i_0)} \gamma_{pi_0-j}) \pmod{\partial L}.$$

It follows that

$$F_{ij} = \gamma_{pi-j} + \sum_{i_0 \notin \{-e, \dots, d-1\}} c_{ii_0} \pi^{\deg(i) - \deg(i_0)} \gamma_{pi_0-j}.$$

If $i_0 \notin \{-e, -(e-1), \dots, d-1\}$, and $i \neq -e$, we have

$$\begin{aligned} \deg(i) - \deg(i_0) + \text{ord}_{\pi}(\gamma_{pi_0-j}) &\geq \deg(i) - \deg(i_0) + \deg(pi_0) - 1 \\ &\geq \deg(pi) + (p-1)(\deg(i_0) - \deg(i)) - 1 \\ &\geq \lfloor \deg(pi) \rfloor + \frac{p-1}{D} - 1 \geq \lfloor \deg(pi) \rfloor + 2. \end{aligned}$$

If $i_0 \notin \{-e, -(e-1), \dots, d-1, d\}$, and $i = -e$, we also have

$$\deg(i) - \deg(i_0) + \text{ord}_{\pi}(\gamma_{pi_0-j}) \geq \lfloor \deg(pi) \rfloor + 2.$$

If $i_0 = d$, and $i = -e$, we have

$$\deg(i) - \deg(i_0) + \text{ord}_{\pi}(\gamma_{pi_0-j}) \geq p.$$

Therefore

$$F_{ij} = \gamma_{pi-j} + \begin{cases} O(\pi^{\lceil \deg(pi) \rceil + 2}), & i \neq -e \\ O(\pi^p), & i = -e. \end{cases}$$

The lemma is proved.

6. GENERIC POLYGON

In this section we prove Theorem 1.9. It follows immediately from the following theorem.

Theorem 6.1. *Let $p \geq 3D$. Then the q -adic Newton polygon of $L(t, f_{\vec{a}}, \mathbb{F}_q)$ coincides with the arithmetic polygon of $[-e, d]$ if and only if $H(\vec{a}) \neq 0$.*

Write

$$\det(1 - s\bar{\Psi}_p \mid L/(\partial L) \text{ over } \mathbb{Q}_p(\pi^{1/D})) = \sum_{i=0}^{b(d+e)} (-1)^i c_i s^i.$$

By Theorems 4.2 and 2.1, Theorem 6.1 follows from the following two theorems.

Theorem 6.2. *Let $p > 3D$. Let $k = 1, 2, \dots, d+e-1$ be such that V_k contains two pairs. Then*

$$\text{ord}_q(c_{bk}) \geq p_{[-e, d]}(k).$$

Theorem 6.3. *Let $p > 3D$. Let $k = 1, 2, \dots, d+e-1$ be such that V_k contains exactly one pair. Then*

$$\text{ord}_q(c_{bk}) \geq p_{[-e, d]}(k)$$

with equality holding if and only if $\bar{H}_k(\vec{a}) \neq 0$.

From now on, we suppose that $q = p^b$, and let ζ be a primitive $(q-1)$ -th roots of unity.

Definition 6.4. *We define the matrix $G = (G_{(i,u),(j,w)})_{-e \leq i, j \leq d-1, 0 \leq u, w \leq b-1}$ by*

$$(\zeta^w)^{\sigma^{-1}} F_{ij}^{\sigma^{-1}} = \sum_{u=0}^{b-1} G_{(i,u),(j,w)} \zeta^u.$$

Lemma 6.5. *We have*

$$\Psi_p(\zeta^w x^j) \equiv \sum_{i=-e}^{d-1} \sum_{u=0}^{b-1} G_{(i,u),(j,w)} \zeta^u x^i \pmod{\partial L}.$$

That is, G is the matrix of $\bar{\Psi}_p$ with respect to the basis over $\mathbb{Q}_p(\pi^{1/D})$ represented by

$$\zeta^u x^i, \quad -e \leq i \leq d-1, 0 \leq u \leq b-1.$$

Proof. Recall that

$$\psi_p \circ E_{\vec{a}}(x^j) \equiv \sum_{i=-e}^{d-1} F_{ij} x^i \pmod{\partial L}.$$

So

$$\Psi_p(\zeta^w x^j) \equiv (\zeta^w)^{\sigma^{-1}} \sum_{i=-e}^{d-1} F_{ij}^{\sigma^{-1}} x^i \pmod{\partial L}.$$

By definition,

$$(\zeta^w)^{\sigma^{-1}} F_{ij}^{\sigma^{-1}} = \sum_{u=0}^{b-1} G_{(i,u),(j,w)} \zeta^u.$$

The lemma now follows.

Corollary 6.6. *We have*

$$\det(1 - s\bar{\Psi}_p \mid L/(\partial L) \text{ over } \mathbb{Q}_p(\pi^{1/D})) = \det(1 - sG).$$

In particular,

$$c_{bk} = \sum_T \det((G_{(i,u),(j,w)})_{(i,u),(j,w) \in T}),$$

where T runs over subsets of

$$\{(i, u) \mid -e \leq i \leq d-1, 0 \leq u \leq b-1\}$$

with cardinality bk .

Lemma 6.7. *Let T_1 and T_2 be two finite sets with equal cardinality. Let g_1 and g_2 be real-valued functions on T_1 and T_2 respectively. Suppose that g_1 and g_2 agree on $T_1 \cap T_2$, and that $g_2(t_2) \geq g_1(t_1)$ for $t_2 \in T_2 \setminus T_1$ and $t_1 \in T_1 \setminus T_2$. Then*

$$\sum_{t \in T_2} g_2(t) \geq \sum_{t \in T_1} g_1(t).$$

Moreover, if $g_2(t_2) > g_1(t_1)$ for $t_2 \in T_2 \setminus T_1$ and $t_1 \in T_1 \setminus T_2$, then the equality holds if and only if $T_1 = T_2$.

Proof. Obvious.

We are now ready to prove Theorem 6.2.

Proof of Theorem 6.2. It suffices to show that, for any subset T of

$$\{(i, u) \mid -e \leq i \leq d-1, 0 \leq u \leq b-1\}$$

with cardinality bk , and any permutation τ of T , we have

$$\text{ord}_\pi\left(\prod_{(i,u) \in T} G_{(i,u),\tau(i,u)}\right) \geq b(p-1)p_{[-e,d]}(k).$$

Let $V_k = \{(m-1, n+1), (m, n)\}$. Then $\frac{n+1}{d} = \frac{m}{e}$. Moreover, the cardinality of the set $\{1 \leq i \leq m-1 \mid pi \equiv m \pmod{e}\}$ is equal to that of $\{1 \leq i \leq n \mid pi \equiv n+1 \pmod{d}\}$. Without loss of generality, we assume that both of them are of cardinality 1. Then

$$(p-1)p_{[-e,d]}(k) = \sum_{i=1}^n \left\lceil \frac{pi-n}{d} \right\rceil + \sum_{i=1}^{m-1} \left\lceil \frac{pi-m+1}{e} \right\rceil + \left\lceil \frac{(p-1)m}{e} \right\rceil - 1.$$

Note that

$$\text{ord}_\pi(G_{(i,u),\tau(i,u)}) = \text{ord}_\pi(F_{i,\tau(i)}).$$

So, if $i > 0$, then

$$\text{ord}_\pi(G_{(i,u),\tau(i,u)}) \geq \begin{cases} \left\lceil \frac{pi-n}{d} \right\rceil, & \tau(i) \leq n, \\ \left\lceil \frac{pi-n}{d} \right\rceil - 1, & \tau(i) > n, \\ \left\lceil \frac{p(n+1)}{d} \right\rceil + 1, & i > n+1. \end{cases}$$

Similarly, if $i < 0$, then

$$\text{ord}_\pi(G_{(i,u),\tau(i,u)}) \geq \begin{cases} \left\lceil \frac{-pi-m+1}{e} \right\rceil, & \tau(i) \geq -m+1, \\ \left\lceil \frac{-pi-m+1}{e} \right\rceil - 1, & \tau(i) \leq -m, \\ \left\lceil \frac{pm}{e} \right\rceil + 1, & i < -m. \end{cases}$$

Therefore

$$\begin{aligned}
\text{ord}_\pi\left(\prod_{(i,u) \in T} G_{(i,u), \tau(i,u)}\right) &\geq \sum_{(i,u) \in T: 1 \leq i \leq n} \left\lceil \frac{pi-n}{d} \right\rceil + \sum_{(-i,u) \in T: 1 \leq i \leq m-1} \left\lceil \frac{pi-m+1}{e} \right\rceil \\
&\quad + \left\lceil \frac{(p-1)m}{e} \right\rceil \sum_{(i,u) \in T: i=n+1 \text{ or } i=-m} 1 + \sum_{(i,u) \in T: i>n+1 \text{ or } i<-m} \left\lceil \frac{pm}{e} \right\rceil \\
&\quad + \sum_{(i,u) \in T: i>n+1 \text{ or } i<-m} 1 - \sum_{(i,u) \in T: \tau(i)>n \text{ or } \leq -m} 1 \\
&\geq \sum_{(i,u) \in T: 1 \leq i \leq n} \left\lceil \frac{pi-n}{d} \right\rceil + \sum_{(-i,u) \in T: 1 \leq i \leq m-1} \left\lceil \frac{pi-m+1}{e} \right\rceil \\
&\quad + \left(\left\lceil \frac{(p-1)m}{e} \right\rceil - 1\right) \sum_{(i,u) \in T: i=n+1 \text{ or } i=-m} 1 + \sum_{(i,u) \in T: i>n+1 \text{ or } i<-m} \left\lceil \frac{pm}{e} \right\rceil.
\end{aligned}$$

By Lemma 6.7, we have

$$\text{ord}_\pi\left(\prod_{(i,u) \in T} G_{(i,u), \tau(i,u)}\right) \geq b\left(\sum_{i=1}^n \left\lceil \frac{pi-n}{d} \right\rceil + \sum_{i=1}^{m-1} \left\lceil \frac{pi-m+1}{e} \right\rceil + \left\lceil \frac{(p-1)m}{e} \right\rceil - 1\right).$$

The proof is completed.

It remains to prove Theorem 6.3.

Lemma 6.8. *Let $p > 3D$. Let $k = 1, 2, \dots, d+e-1$ be such that $V_k = \{(m, n)\}$. Then*

$$c_{bk} = \det((G_{(i,u), (j,w)})_{-m \leq i, j \leq n, 0 \leq u, w \leq b-1}) + O(\pi^{b(p-1)p_{[-e,d]}(k) + \frac{1}{D}}).$$

Proof. It suffices to show that, for any subset T of

$$\{(i, u) \mid -e \leq i \leq d-1, 0 \leq u \leq b-1\}$$

with cardinality bk which is different from $\{-m, \dots, n\} \times \{0, \dots, b-1\}$, and any permutation τ of T , we have

$$\text{ord}_\pi\left(\prod_{(i,u) \in T} G_{(i,u), \tau(i,u)}\right) > b(p-1)p_{[-e,d]}(k).$$

First we suppose that $I_k = \{(m, n)\}$. Note that, if $i > 0$, then

$$\text{ord}_\pi(G_{(i,u), \tau(i,u)}) \geq \begin{cases} \left\lceil \frac{pi-n}{d} \right\rceil, & \tau(i) \leq n, \\ \left\lceil \frac{pi-n}{d} \right\rceil - 1, & \tau(i) > n, \\ \left\lceil \frac{pm}{d} \right\rceil + \frac{p}{d} - 2, & i > n. \end{cases}$$

Similarly, if $i < 0$, then

$$\text{ord}_\pi(G_{(i,u), \tau(i,u)}) \geq \begin{cases} \left\lceil \frac{-pi-m}{e} \right\rceil, & \tau(i) \geq -m, \\ \left\lceil \frac{-pi-m}{e} \right\rceil - 1, & \tau(i) < -m, \\ \left\lceil \frac{pm}{e} \right\rceil + \frac{p}{e} - 2, & i < -m. \end{cases}$$

So

$$\text{ord}_\pi\left(\prod_{(i,u) \in T} G_{(i,u), \tau(i,u)}\right) \geq \sum_{(i,u) \in T: 1 \leq i \leq n} \left\lceil \frac{pi-n}{d} \right\rceil + \sum_{(-i,u) \in T: 1 \leq i \leq m} \left\lceil \frac{pi-m}{e} \right\rceil$$

$$\begin{aligned}
 & + \sum_{(i,u) \in T: i > n} \left\lceil \frac{pn}{d} \right\rceil + \sum_{(-i,u) \in T: i > m} \left\lceil \frac{pm}{e} \right\rceil \\
 & + \sum_{(i,u) \in T: i > n \text{ or } i < -m} \left(\frac{p}{D} - 2 \right) - \sum_{(i,u) \in T: \tau(i) > n \text{ or } < -m} 1 \\
 & > \sum_{(i,u) \in T: 1 \leq i \leq n} \left\lceil \frac{pi - n}{d} \right\rceil + \sum_{(-i,u) \in T: 1 \leq i \leq m} \left\lceil \frac{pi - m}{e} \right\rceil \\
 & + \sum_{(i,u) \in T: i > n} \left\lceil \frac{pn}{d} \right\rceil + \sum_{(-i,u) \in T: i > m} \left\lceil \frac{pm}{e} \right\rceil
 \end{aligned}$$

By Lemma 6.7, we have

$$\text{ord}_\pi \left(\prod_{(i,u) \in T} G_{(i,u), \tau(i,u)} \right) > b(p-1)p_{[-e,d]}(k).$$

Secondly, we suppose that I_k contains two pairs. Without loss of generality, we may assume that $I_k = \{(m, n), (m+1, n-1)\}$. Then $\frac{m+1}{e} = \frac{n}{d}$,

$$pi \not\equiv m+1 \pmod{e}, \quad 1 \leq i \leq m,$$

and there is exactly one $1 \leq i \leq n-1$ such that

$$pi \equiv n \pmod{d}.$$

So

$$(p-1)p_{[-e,d]}(k) = \sum_{i=1}^{n-1} \left\lceil \frac{pi - n + 1}{d} \right\rceil + \sum_{i=1}^m \left\lceil \frac{pi - m - 1}{e} \right\rceil + \left\lceil \frac{(p-1)n}{d} \right\rceil - 1.$$

Note that, if $i > 0$, then

$$\text{ord}_\pi(G_{(i,u), \tau(i,u)}) \geq \begin{cases} \left\lceil \frac{pi - n + 1}{d} \right\rceil, & \tau(i) \leq n-1, \\ \left\lceil \frac{pi - n + 1}{d} \right\rceil - 1, & \tau(i) \geq n, \\ \left\lceil \frac{pn}{d} \right\rceil + \frac{p}{d} - 2, & i > n. \end{cases}$$

Similarly, if $i < 0$, then

$$\text{ord}_\pi(G_{(i,u), \tau(i,u)}) \geq \begin{cases} \left\lceil \frac{-pi - m - 1}{e} \right\rceil, & \tau(i) \geq -m-1, \\ \left\lceil \frac{-pi - m - 1}{e} \right\rceil - 1, & \tau(i) < -m-1, \\ \left\lceil \frac{pn}{d} \right\rceil + \frac{p}{e} - 2, & i < -m-1. \end{cases}$$

So

$$\begin{aligned}
 \text{ord}_\pi \left(\prod_{(i,u) \in T} G_{(i,u), \tau(i,u)} \right) & \geq \sum_{(i,u) \in T: 1 \leq i < n} \left\lceil \frac{pi - n + 1}{d} \right\rceil + \sum_{(-i,u) \in T: 1 \leq i \leq m} \left\lceil \frac{pi - m - 1}{e} \right\rceil \\
 & + \left\lceil \frac{(p-1)n}{d} \right\rceil \sum_{(i,u) \in T: i=n \text{ or } i=-m-1} 1 + \sum_{(i,u) \in T: i > n \text{ or } i < -m-1} \left\lceil \frac{pn}{d} \right\rceil \\
 & + \sum_{(i,u) \in T: i > n \text{ or } i < -m-1} \left(\frac{p}{D} - 2 \right) - \sum_{(i,u) \in T: \tau(i) \geq n \text{ or } < -m-1} 1.
 \end{aligned}$$

If $\{(i, u) \in T : i > n \text{ or } i < -m - 1\} \neq \emptyset$, then

$$\begin{aligned} \text{ord}_\pi \left(\prod_{(i,u) \in T} G_{(i,u), \tau(i,u)} \right) &> \sum_{(i,u) \in T: 1 \leq i < n} \left\lceil \frac{pi - n + 1}{d} \right\rceil + \sum_{(-i,u) \in T: 1 \leq i \leq m} \left\lceil \frac{pi - m - 1}{e} \right\rceil \\ &+ \left(\left\lceil \frac{(p-1)n}{d} \right\rceil - 1 \right) \sum_{(i,u) \in T: i=n} 1 + \left\lceil \frac{(p-1)n}{d} \right\rceil \sum_{(i,u) \in T: i=-m-1} 1 + \sum_{(i,u) \in T: i > n \text{ or } i < -m-1} \left\lceil \frac{pn}{d} \right\rceil. \end{aligned}$$

By Lemma 6.7, we have

$$\text{ord}_\pi \left(\prod_{(i,u) \in T} G_{(i,u), \tau(i,u)} \right) > \sum_{i=1}^{n-1} \left\lceil \frac{pi - n + 1}{d} \right\rceil + \sum_{i=1}^m \left\lceil \frac{pi - m - 1}{e} \right\rceil + \left\lceil \frac{(p-1)n}{d} \right\rceil - 1.$$

If $\{(i, u) \in T : i > n \text{ or } i < -m - 1\} = \emptyset$, then

$$\begin{aligned} \text{ord}_\pi \left(\prod_{(i,u) \in T} G_{(i,u), \tau(i,u)} \right) &\geq \sum_{(i,u) \in T: 1 \leq i < n} \left\lceil \frac{pi - n + 1}{d} \right\rceil + \sum_{(-i,u) \in T: 1 \leq i \leq m} \left\lceil \frac{pi - m - 1}{e} \right\rceil \\ &+ \left(\left\lceil \frac{(p-1)n}{d} \right\rceil - 1 \right) \sum_{(i,u) \in T: i=n} 1 + \left\lceil \frac{(p-1)n}{d} \right\rceil \sum_{(i,u) \in T: i=-m-1} 1. \end{aligned}$$

By Lemma 6.7, we also have

$$\text{ord}_\pi \left(\prod_{(i,u) \in T} G_{(i,u), \tau(i,u)} \right) > \sum_{i=1}^{n-1} \left\lceil \frac{pi - n + 1}{d} \right\rceil + \sum_{i=1}^m \left\lceil \frac{pi - m - 1}{e} \right\rceil + \left\lceil \frac{(p-1)n}{d} \right\rceil - 1.$$

The proof is completed.

Definition 6.9. We write $\alpha \sim \beta$ to mean that $\alpha = u\beta$ for some p -adic unit u .

Theorem 6.10. Let $p > 3D$. Let $k = 1, 2, \dots, d + e - 1$ be such that $V_k = \{(m, n)\}$. Then

$$c_{bk} \sim \det((F_{ij})_{-m \leq i, j \leq n})^b + O(\pi^{b(p-1)p_{[-e, d]}(k) + \frac{1}{D}}).$$

Proof. It suffices to show that

$$\det((G_{(i,u), (j,w)})_{-m \leq i, j \leq n, 0 \leq u, w \leq b-1}) \sim \det((F_{ij})_{-m \leq i, j \leq n})^b.$$

Let $V = \bigoplus_{i=-m}^n \mathbb{Q}_q(\pi^{1/D})e_i$ be a k -dimensional vector space over $\mathbb{Q}_q(\pi^{1/D})$ with standard basis e_{-m}, \dots, e_n . Let $F = (F_{ij})_{-m \leq i, j \leq n}$ act on it in the standard way, and let σ act on it coordinate-wise. Then

$$\sigma^{-1} \circ F(\zeta^w e_j) = (\zeta^w)^{\sigma^{-1}} \sum_{i=-m}^n F_{ij}^{\sigma^{-1}} e_i.$$

Therefore, G is the matrix of $\sigma^{-1} \circ F$ on V with respect to the basis over $\mathbb{Q}_p(\pi^{1/D})$:

$$\zeta^u e_i, \quad -m \leq i \leq n, 0 \leq u \leq b-1.$$

As σ is just a re-ordering of the basis, we have

$$\det((G_{(i,u), (j,w)})_{-m \leq i, j \leq n, 0 \leq u, w \leq b-1}) \sim \det((F_{ij})_{-m \leq i, j \leq n})^b.$$

The theorem is proved.

Lemma 6.11. *Let $p > 3D$. Let $k = 1, 2, \dots, d + e - 1$ be such that $V_k = \{(m, n)\}$. Then*

$$\det((F_{ij})_{-m \leq i, j \leq n}) = \sum_{\tau \in S_k} \text{sgn}(\tau) \prod_{i=-m}^n F_{i, \tau(i)} + O(\pi^{(p-1)p_{[-e, d]}(k)+1/D}).$$

Proof. For $j \leq n$, we have

$$\lceil \frac{pi - j}{d} \rceil = \lceil \frac{pi - n + (n - j)}{d} \rceil \geq \lceil \frac{pi - n}{d} \rceil$$

with equality holding if and only if

$$j \geq n - d\{-\frac{pi - n}{d}\}.$$

Similarly, for $j \geq -m$, we have

$$\lceil \frac{-pi + j}{e} \rceil = \lceil \frac{-pi - m + (m + j)}{e} \rceil \geq \lceil \frac{-pi - m}{e} \rceil$$

with equality holding if and only if

$$j \leq -m + e\{\frac{pi + m}{e}\}.$$

So, if $\tau \notin S_k$ is a permutation of $\{-m, -(m-1), \dots, n\}$, then

$$\begin{aligned} \text{ord}_\pi(\prod_{i=-m}^n F_{i, \tau(i)}) &\geq \sum_{i=-m}^n \lceil \deg(pi - \tau(i)) \rceil \\ &\geq 1 + \sum_{i=1}^n \lceil \frac{pi - n}{d} \rceil + \sum_{i=1}^m \lceil \frac{pi - m}{e} \rceil. \end{aligned}$$

Hence

$$\det((F_{ij})_{-m \leq i, j \leq n}) = \sum_{\tau \in S_k} \text{sgn}(\tau) \prod_{i=-m}^n F_{i, \tau(i)} + O(\pi^{(p-1)p_{[-e, d]}(k)+1/D}).$$

The lemma is proved.

We are now ready to prove Theorem 6.3. By the above lemmas, it suffices to prove the following theorem.

Theorem 6.12. *Let $p > 3D$. Let $k = 1, 2, \dots, d + e - 1$ be such that $V_k = \{(m, n)\}$. Then*

$$\det((F_{ij})_{-m \leq i, j \leq n}) = \pi^{(p-1)p_{[-e, d]}(k)} \hat{a}_d^{u_k} \hat{a}_{-e}^{v_k} H_k(\vec{\hat{a}}) + O(\pi^{(p-1)p_{[-e, d]}(k)+1/D}),$$

where $\vec{\hat{a}} = (\hat{a}_{-e}, \dots, \hat{a}_d)$, and u_k, v_k are integers depending on k .

Proof. By Lemmas 6.11 and 5.5, we have

$$\det((F_{ij})_{-m \leq i, j \leq n}) = \sum_{\tau \in S_k} \text{sgn}(\tau) \prod_{i=-m}^n \gamma_{pi - \tau(i)} + O(\pi^{(p-1)p_{[-e, d]}(k)+1/D}).$$

By Lemmas 5.2 and 5.3, we have

$$\gamma_{pi - \tau(i)} = \begin{cases} \pi^{\lceil \frac{pi - n}{d} \rceil} \lambda_{\lfloor \frac{pi - \tau(i)}{d} \rfloor} \lambda_{\lceil \frac{pi - \tau(i)}{d} \rceil} \hat{a}_d^{\lceil \frac{pi - n}{d} \rceil - 1} \hat{a}_{r_i - \tau(i)} + O(\pi^{\lceil \frac{pi - n}{d} \rceil}), & i > 0 \\ \pi^{\lceil \frac{-pi - m}{e} \rceil} \lambda_{\lfloor \frac{-pi + \tau(i)}{e} \rfloor} \lambda_{\lceil \frac{-pi + \tau(i)}{e} \rceil} \hat{a}_{-e}^{\lceil \frac{-pi - m}{e} \rceil - 1} \hat{a}_{-r_i - \tau(i)} + O(\pi^{\lceil \frac{-pi - m}{e} \rceil + 1}), & i < 0. \end{cases}$$

The theorem now follows.

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